Jones — A critique of the variational method in scattering problems

NYU EM 78

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-78

A Critique of the Variational Method in Scattering Problems

D. S. JONES

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

CONTRACT NO. AF-19(122)-42

MAY 1955

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. EM-78

A CRITIQUE OF THE VARIATIONAL METHOD IN SCATTERING PROBLEMS

D. S. Jones

D S Jones
D. S. Jones

Morris Kline
Morris Kline
Project Director

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

The research reported in this document has been made possible through support and sponsorship extended by the Air Force Cambridge Research Center, under Contract No. AF 19(122)-42. It is published for technical information only and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1955

Abstract

It is shown that the variational method of dealing with the integral equations of scattering problems is equivalent to solving the integral equation directly by Galerkin's method and using the standard formula for the amplitude of the scattered wave. The second method also satisfies the reciprocity theorem. It is therefore suggested that the reciprocity theorem be used as the basis of approximation without the introduction of variational formulas.

The error involved in using an approximate solution is discussed and it is shown that only a special set of approximations can lead to accuracy at low frequencies. Some ways in which bounds for the error may be obtained in special problems are also given.

Table of Contents

Page

1. Introduction 1
2. The equivalence of the variational method and Galerkin's method 2
3. The accuracy of the approximation 7
4. The low frequency behavior 10
5. Bounds for the error 12

References 18

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

1. Introduction

The fact that an integral equation can be formulated as a variational principle was stated as early as 1884 by Volterra[1] but it has not been used for practical calculation in scattering problems until recent years when it was discovered by Schwinger that the amplitude of the scattered field is closely related to the quantity whose variation has to be considered. (In connection with small-amplitude sound waves, see Levine and Schwinger[2], and in connection with atomic collisions see Schwinger's results as quoted by Kohn[3]). The essence of the variational method is that, if a good approximation for the field is inserted in the variational expression, an improved approximation for the scattered amplitude should result. One particular way of choosing the approximation, that given by Levine and Schwinger[2], is to expand the field in a set of functions and solve the simultaneous linear equations resulting from the variational principle for the coefficients. It will be shown in the following that this is exactly equivalent to Galerkin's method of solving the integral equation and, furthermore, that only by expanding the field in a special set of functions can accuracy be obtained. Also this solution satisfies the reciprocity theorem.

Another way of approximation is to choose for the field an expression which is physically plausible and/or mathematically simple and insert it immediately in the variational expression. It will be shown that this is equivalent to choosing an approximation which satisfies the reciprocity theorem.

It appears, therefore, that fundamentally the use of the variational method is to ensure that the reciprocity theorem is complied with. Since the reciprocity theorem is easily written down, it is obviously simpler to formulate our analysis in terms of it than to introduce the variational principle. The advantage is mainly one of clarity for, once the equation(s) to determine the approximation have been obtained, the labor required to calculate the scattered amplitude is the same by either method.

2. The equivalence of the variational method and Galerkin's method[†]

The mathematical problem encountered in scattering may be stated as follows: given a function f and operator L, find a function g such that

$$f = Lg \quad . \tag{1}$$

The function f is determined by the incident field and g can be regarded as a suitable source distribution. The equation is supposed to have a unique solution. Examples of equation (1) are

$$f(\underline{r}') = \int_S g(\underline{r})\, \Psi(\underline{r}, \underline{r}')\, d\underline{r}$$

for one of the boundary conditions in diffraction by an aperture, and

$$f(\underline{r}') = g(\underline{r}') + \int_S V(\underline{r}) g(\underline{r}) \Psi(\underline{r},\underline{r}') d\underline{r}$$

for atomic collisions or scattering by a dielectric obstacle. In these equations

$$\Psi(\underline{r}, \underline{r}') = \frac{e^{-ik|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|} \quad ,$$

S is the area of the aperture, T is the volume of the scatterer and V is the potential of the scattering particle. (For the dielectric obstacle V is essentially a measure of the dielectric constant.)*

Often it is the scattered field at some distance from the obstacle and not g itself which is of primary concern. The amplitude A of the scattered field may be written

---

[†] Some results similar to those of this section have been obtained independently by Lippmann[4].

* Eq. (1) and the following analysis are applicable to vector field problems on regarding f and g as vectors and L as a vector operator but no specific example will be quoted.

(2) $$A = (f', g) = (g, f')$$

where f' is some suitable incident field and (f', g) means the inner product of f' and g. For example, the amplitude of the far field in the direction of the unit vector $\underline{n}$ is obtained correctly if we define (f', g) by

$$(f',g) = \int_S f'(\underline{r})g(\underline{r})d\underline{r}$$

for the aperture problem or by

$$(f', g) = \int_T f'(\underline{r})V(\underline{r})g(\underline{r})d\underline{r}$$

for the atomic collision and take $f' = e^{ik\underline{r}\cdot\underline{n}}$. (A different choice of f' would lead to some other quantity, the near field for example, but would not invalidate the following analysis provided that it was a possible incident field.)

Let g' be such that

(3) $$f' = Lg' .$$

Assume that L is a symmetric operator such that

(4) $$(h_1, Lh_2) = (h_2, Lh_1) .$$

Then, from (3), (4) and (1),

(5)
$$\begin{aligned}(f', g) &= (Lg', g)\\ &= (g', Lg)\\ &= (g', f)\end{aligned}$$

or

$$A = (f', g) = (f, g') = (g', Lg) .$$

This is the reciprocity theorem which exists for most scattering problems.

By an obvious use of this theorem, we can write

$$A = \frac{(f',g)(f,g')}{(g, Lg')} \quad . \tag{6}$$

It can be confirmed that the necessary and sufficient condition for (1) and (3) to be satisfied is that this expression for A is stationary for independent small variations of g and g' about their correct values. Other variational principles are available (see, for example, Kohn[3]) but will not be considered here.

To demonstrate that the choice of an approximation to make (6) stationary is equivalent to Galerkin's solution of (1) we shall consider first the special case in which $f = f'$ so that $g = g'$. This corresponds, for example, to the problem of the circular aperture in a normally incident plane sound wave with the scattering observed in the direction of propagation of the incident wave. In this special case

$$A = \frac{(f,g)^2}{(g,Lg)} \quad . \tag{7}$$

Let G_o, G_1,... be a set of functions each of which is independent of the others and each of which satisfies any conditions, e.g., edge conditions, imposed on g. If an infinite number of G_o, G_1... is used, the set is also required to be complete. Now try an approximation G to g given by

$$G = \sum_n c_n G_n \tag{8}$$

where the sum may involve either a finite or infinite number of terms and the constants c_n are to determined. We shall now prove the following

Theorem 1 The necessary and sufficient conditions for the expression in (7) to be stationary for small variations of G are

$$(f, G_n) = (G_n, LG) \qquad (n = 0,1,\ldots). \tag{9}$$

G then satisfies $(f,G) = (G, LG)$ and $A = (f, G)$.

The importance of this theorem lies in the fact that equations (9) are precisely those obtained by substituting the expansion (8) in (1) and taking the inner product of both sides with G_n, which is Galerkin's process. Then A may be calculated directly from its original definition (2) because the approximation satisfies the reciprocity theorem. Obviously this direct method of setting up the equations is to be preferred to determining them via the variation principle. It should be remarked that, if we make a one-term approximation so that $G = c_o G_o$, the reciprocity theorem is satisfied if $(f, G_o) = c_o(G_o, LG_o)$ and (f, G) becomes $(f, G_o)^2/(G_o, LG_o)$ which is the same as would be calculated from (7).

We proceed now to prove the theorem. The first variation of (7) with $g = G$ when c_n becomes $c_n + \delta c_n$ is

$$\frac{\left\{2(f,G)\left(f, \sum_n G_n \delta c_n\right)(G, LG) - 2(f,G)^2 \left(LG, \sum_n G_n \delta c_n\right)\right\}}{(G, LG)^2} . \tag{10}$$

Suppose now that equations (9) are satisfied. Then (10) vanishes if $(f, G) = (G, LG)$. Now $(f,g) = \left(f, \sum_n c_n G_n\right) = \left(LG, \sum_n c_n G_n\right)$ on account of (9); hence

$$(f, G) = (G, LG) \tag{11}$$

so that the first variation is zero. The equations (9) are thus sufficient for (7) to be stationary.

Suppose now that the first variation is zero for independent δc_n. Then

$$(G, LG)(f, G_n) = (f,G)(LG, G_n) \qquad (n = 0, 1, \ldots).$$

These equations are homogeneous in the constants $c_o, c_1 \ldots$. Hence if there is one solution there are any number of the form $Cc_o, Cc_1 \ldots$ where C is a constant. There is therefore a solution in which

$$(G, LG) = (f, G) \,. \tag{12}$$

Such a solution satisfies equations (9) and also the reciprocity theorem. The necessity for the equations has therefore been shown.

On account of (11) and (12), $A = (f,G)$ and the theorem has been proved.

The corresponding theorem for the more general expression (6) will now be stated. No proof will be given since it runs parallel to that of Theorem 1. We take G' to be the approximation to g' and try

$$G' = \sum_m c_m G'_m ,$$

this series containing as many terms as (8). Then

Theorem 2 The necessary and sufficient conditions for the expression (6) with $g = G$ and $g' = G'$ to be stationary for small independent variations of G and G' are

$$(f, G'_m) = (LG, G'_m) \qquad (m = 0, 1, \ldots)$$

and

$$(f', G_m) = (LG', G_m) \,.$$

G and G' satisfy the reciprocity relation $(f', G) = (G, LG') = (f, G')$. The value of A is either (f, G') or (f', G).

3. The accuracy of the approximation

To determine a good approximation it is necessary to specify precisely what is meant by good. This involves stipulating the behavior of the operators and functions in some manner. We shall choose the behavior which is appropriate to a perturbation. Attention will be concentrated on the case $f = f'$ so that $A = (f, g)$ and the reciprocity theorem takes the form $(f, g) = (Lg, g)$.

Let α be a parameter such that

(i) for some range of α

$$L = L_0 + \alpha L_1 + \alpha^2 L_2 + \dots$$

$$f = f_0 + \alpha f_1 + \alpha^2 f_2 + \dots$$

$$g = g_0 + \alpha g_1 + \alpha^2 g_2 + \dots$$

$$G = G^0 + \alpha G^1 + \alpha^2 G^2 + \dots ,$$

(ii) the only solution of $L_0 h = 0$ is $h = 0$,

(iii) $(h_1, L_p h_2) = (L_p h_1, h_2)$ for all p.

When α is small we are writing the operator L as one which is slightly perturbated from L_0. If α measures the ratio of the linear dimensions of the obstacle and the wavelength, and L_0 is the static operator, then we are considering the low-frequency behavior. It has been shown by Magnus[5] that assumption (i) is justified for the problem of a low-frequency plane wave falling on a circular aperture.

If we substitute the formulas of (i) in (1) and equate the coefficients of powers of α the equations to determine g are

$$\sum_{r=0}^{p} L_{p-r}\, g_r = f_p \qquad (p = 0, 1, \dots) . \tag{13}$$

If the inverse of the operator L_0 is known, g can be found, in principle, correct to any power of α. Assumption (ii) ensures a unique solution.

The approximation G is to be determined so that the reciprocity theorem is satisfied, i.e., so that $(G, LG) = (f,G)$. Equating the coefficients of the like powers of α in the expansions of both sides, we obtain

$$\sum_{r=0}^{p} \sum_{s=0}^{p-r} \left(G^r, L_{p-r-s} G^s\right) = \sum_{r=0}^{p} \left(G^r, f_{p-r}\right) \qquad (p = 0,1,\ldots)$$

or

$$\sum_{r=0}^{p} \sum_{s=0}^{p-r} \left(G^r, L_{p-r-s} G^s - L_{p-r-s}\, g_s\right) = 0 \tag{14}$$

on account of (13).

Now we are going to approximate (f,g) by (f,G). Both quantities can be expanded in powers of α and the coefficients of α^p are the same if

$$\sum_{s=0}^{p} \left(f_{p-s}, G^s - g_s\right) = 0 \,. \tag{15}$$

Consequently the first P terms of (f,g) are given correctly by the approximation if (15) holds for all $p \leq P - 1$.

Let us now suppose that $G^m = g_m$ $(m \leq M)$. Then, in (14), we can replace G^r by g_r in all the inner products in which $p - r \leq M$ for in these the second member is identically zero. Also G^r may be replaced by g_r in all inner products in which $r \leq M$. By this means G^r is replaced by g_r in all inner products in (14) provided that $p \leq 2M + 1$. Hence, after a use of assumption (iii),

$$\sum_{r=0}^{p} \sum_{s=0}^{p-r} \left(L_{p-r-s}\, g_r,\, G^s - g_s\right) = 0 \qquad (p \leq 2M + 1)$$

or

$$\sum_{s=0}^{p} \sum_{r=0}^{p-s} \left(L_{p-r-s}\, g_r,\, G^s - g_s\right) = 0 \qquad (p \leq 2M + 1)$$

on interchanging the order of summation. This equation becomes, on taking account of (13),

$$\sum_{s=0}^{p} \left(f_{p-s},\, G^s - g_s\right) = 0 \qquad (p \leq 2M + 1)$$

which is the same as (15). Thus if $G^m = g_m$ $(m \leq M)$ the first 2M + 2 terms of (f,g) are given correctly by the approximation.

If we carry out a similar analysis with p = 2M + 2 we find

$$\sum_{s=0}^{2M+2} \left(f_{2M+2-s},\, G^s - g_s\right) + \left(G^{M+1} - g_{M+1},\, L_o G^{M+1} - L_o g_{M+1}\right) = 0$$

so that the (2M+3)rd term of (f,g) is given correctly only if

$$\left(G^{M+1} - g_{M+1},\, L_o G^{M+1} - L_o g_{M+1}\right) = 0 . \tag{16}$$

This result enables us to obtain some information which is particularly applicable to the low-frequency behavior. At low frequencies, where L_o is the static operator and α is purely imaginary, the following additional assumptions hold in most scattering problems:

(iv) L_p operating on a real function produces a real function for all p;

(v) if h is real and $(h, L_o h) = 0$ then h = 0;

(vi) f_p is purely real for all p.

With assumptions (iv) and (vi) it is easy to see that all g_p and G^p are real. Hence if (16) holds, assumption (v) shows that $G^{M+1} = g_{M+1}$. In particular, this applies if $M = -1$.

Consequently we have shown:

Theorem 3 The sufficient conditions for (f,G), G satisfying the reciprocity relation, to give correctly the first $2M+2$ terms of the power series expansion of (f,g) are

$$G^m = g_m \qquad\qquad (m \leq M)$$

when assumptions (i) to (iii) are satisfied. If, in addition, assumptions (iv) to (vi) hold, the conditions are also necessary.

The corresponding theorem for the more general case is not so powerful. It is

Theorem 4 The sufficient conditions for (f',G) or (f,G'), G and G' satisfying the reciprocity relations, to give correctly the first $2M+2$ terms of the power series expansion of A are

$$G^m = g_m, \quad G'^m = g'_m \qquad\qquad (m \leq M)$$

when assumptions (i) to (iii) hold with appropriate addition to (i) for G' and g'.

The proof is similar to that for Theorem 3 and will not be given.

4. The low-frequency behavior

In this section we shall consider the case $f = f'$ when assumptions (i) to (vi) hold. These assumptions are specially applicable to the low-frequency behavior and we shall make particular reference to the process employed by Levine and Schwinger[2].

Assume that $G = \sum_{m=0}^{n} c_m G_m$ where the G_m are real, independent of α and defined in a similar way to those of Theorem 1. The constants c_m are determined by equations (9). One hopes to obtain an approximation which is reasonable over the whole frequency band by a suitable choice of G_m. By means of Theorem 3 we can say what this choice must be for the low-frequency end. While we cannot guarantee from this work that the choice will also be appropriate for the high frequency end, we can say that, at any rate, it is not precluded.

Suppose that we desire that this approximation G gives the first $2n + 2N + 2$ terms of (f,g) correctly at low frequencies, N being an integer. If, in particular, $n = 0$ we want to make N as large as possible. Now, when the first $2n + 2N + 2$ terms of (f,g) are correct, it follows from Theorem 3 that $G^m = g_m$ $(m \leq n + N)$. But G^m can be expressed in terms of $G_o, G_1, \dots, G_n$. Hence the first $2n + 2N + 2$ terms of (f,g) can be correct only if $G_o, G_1, \dots, G_n$ are linear combinations of $g_o, g_1, \dots, g_{n+N}$ or, alternatively, if $g_o, g_1, \dots, g_{n+N}$ are linear combinations of $G_o, G_1, \dots, G_n$.

If, in addition, we require the first $2n + 2N + 2$ terms to be correct for all n then, on taking $n = 0$, $g_o, g_1, \dots, g_N$ must be constant multiples of G_o. Taking $n = 1$, we deduce that g_{N+1} must be a linear combination of G_o and G_1. From $n = 2$, g_{N+2} must be a linear combination of G_o, G_1 and G_2 and the process continues.

Thus the maximum number of terms which can be obtained by a one-element approximation to G $(n = 0)$ is determined by the first g_n which is not a constant multiple of g_o.

Consider now the application of these results to the problem considered by Levine and Schwinger. In that problem $f_1 = 0$, $L_1 \equiv 0$, so that $g_1 = 0$, but g_2 is not a constant multiple of g_o. Hence $N = 1$ and the maximum number of terms of (f,g) to be obtained from the approximation $G = c_o G_o$ is 4 and can be obtained only by choosing $G_o = g_o$. From $G = c_o G_o + c_1 G_1$ we can obtain 6 terms provided that G_1 is a linear combination of g_o and g_2. Similarly, a suitable choice of G_2 will give 8 terms of (f,g) and so on. The choice of G_n made by Levine and Schwinger is in fact the one dictated by the above accuracy considerations. Of course it is not necessary to make their particular choice of G_1 to ensure low-frequency accuracy; we can choose any linear combinations of g_o and g_2 so long as both are included. Similar remarks apply to any G_n and, in general, we would choose those that are most convenient for dealing with equations (9).

Finally, there is one thing to be careful of in the use of Theorem 3. The statement of accuracy in this theorem depends upon L and therefore upon the accuracy with which L is approximated. Thus, when the inner products in (9) have to be evaluated approximately (as often happens), care must be taken that this evaluation does not vitiate Theorem 3. Only if the constants in (9) are obtained correct to $O\left(\alpha^{2M+1}\right)$ at the low-frequency end is Theorem 3 available.

5. Bounds for the error

Although the analysis of the preceding section indicates how the trial function must be chosen, it does not give a precise estimate of the error nor does it state the range of α for which the approximation is good. In practice, it may well be more desirable to use an approximate field for which it is possible to say that (f,g) is estimated correctly to (say) 10% for a certain range of wavelength than to insist that the field satisfies the reciprocity theorem and gives (say) 1% accuracy over an unknown range. We shall now show how an estimate of the error can be made in certain circumstances.

Let $\gamma = g - G$ where G is some approximation to g. Then

$$(f,\gamma) = (Lg,\gamma) = (g,L\gamma)$$

or

$$(f,g) = (f,G) + (G,L\gamma) + (\gamma,L\gamma) . \tag{17}$$

Since

$$(18) \qquad L\gamma = f - LG$$

the second term may also be written as $(G, f-LG)$. It is easy to see from (17) that $2(f,g) - (g,Lg)$ is a stationary expression for (f,g).

If G satisfies the reciprocity theorem the second term in (17) is zero and the error made by the approximation is $(\gamma, L\gamma)$. This result sometimes enables us to state the sign of the error. For example, if all quantities are real and L is positive definite, we see that (f,G) is a lower bound to (f,g).

Even if G does not satisfy the reciprocity theorem the only term which cannot be calculated in (17) is the third because the second can always be evaluated by means of (18). An estimate of the goodness of the approximation therefore requires only an estimate of the third term in (17); this estimate depends essentially on being able to deduce properties of γ from a knowledge of $L\gamma$. There does not appear to be any method which we can say will be generally applicable because the operator and functions occurring are usually complex but we shall give two methods which are available in suitable circumstances.

Firstly, suppose that it is possible to say

$$|(h_1,h_2)|^2 \leq (h_1,h_1^*)(h_2,h_2^*)$$

where the star indicates a complex conjugate and suppose, further, that there is a $\mu > 0$ such that

$$(L\gamma, L^*\gamma^*) \geq \mu(\gamma,\gamma^*)$$

for all γ which can arise. Then

$$|(\gamma, L\gamma)| \leq \frac{1}{\sqrt{\mu}} (L\gamma, L^*\gamma^*) .$$

Since μ and $L\gamma$ ar both known this inequality gives an estimate of the error. We may also determine the accuracy of the other approximations. For $(f', g) = (f', G) + (f', \gamma)$ and

$$(19) \qquad |(f',\gamma)|^2 \leq \frac{1}{\mu} (f', f'^*)(L\gamma, L^*\gamma^*) ;$$

all quantities are known on the right hand side of (19).

A second method, related to that used by Kato[6] in problems of atomic collisions, depends upon adjusting the incident field until a prescribed scattered amplitude is obtained. The scattered amplitude corresponding to the original incident field can then be deduced immediately. Let ν be a complex constant such that, if

$$(20) \qquad \nu f = L g^o,$$

$$(21) \qquad (f, g^o) = 1 .$$

Since $g^o = \nu g$ we see that $(f,g) = \frac{1}{\nu}$.

Consider now G chosen so that

$$(22) \qquad (f, G) = 1 .$$

Let $G^o = \nu' f - (L-1) G = \nu' f - MG$ where ν' is a complex constant. G^o is considered as an approximation to g^o. Then

$$(g^o, G - G^o) = (g^o, LG - \nu' f) = (Lg^o, G) - \nu'$$
$$= \nu - \nu'$$

on account of (20, (21) and (22). Hence

$$(23) \qquad \nu = \nu' + (G^o, G - G^o) + (g^o - G^o, G - G^o) .$$

We can regard the first two terms as giving an approximation to ν, the third being the error.

To estimate this error, we introduce the equation

$$(24) \qquad (1 + \lambda)\, M \phi + \phi = \Lambda f$$

where $(1 + \lambda)(f, \phi) = 0$ and Λ is a non-zero constant to be chosen later. We assume that non-zero ϕ can be obtained only for a discrete set of real λ of which λ_n is a typical one, ϕ_n being the corresponding solution. Since

$$(\phi_n, \phi_m) = -(\lambda_m + 1)(\phi_n, M \phi_m) = -(\lambda_m + 1)(M\phi_n, \phi_m) = \frac{1 + \lambda_m}{1 + \lambda_n} (\phi_n, \phi_m)$$

we choose Λ for each λ_n so that

$$(\phi_m, \phi_n) = \delta_{nm}$$

where $\delta_{nm} = 0$ $(n \neq m)$, $= 1 (n = m)$. It will be assumed that the ϕ_n form a complete set.

Define the constants a_n and b_n by

$$(G - G^o, \phi_n) = a_n$$

and

$$(g^o - G^o, \phi_n) = b_n \quad .$$

Then

$$b_n = (\nu f - M g^o - \nu' f + M G, \phi_n)$$

$$= (M (G - g^o), \phi_n)$$

because of (24); thus

$$b_n = (G - g^0, M \phi_n)$$

$$= - \frac{1}{1 + \lambda_n} (G - g^0, \phi_n)$$

on account of (21) and (22). Also

$$a_n = (LG - Lg^0, \phi_n)$$

$$= (G - g^0, L \phi_n)$$

$$= \frac{\lambda_n}{1 + \lambda_n} (G - g^0, \phi_n)$$

$$= -\lambda_n b_n .$$

Therefore

$$(g^0 - G^0, G - G^0) = \sum a_n b_n = -\sum \frac{1}{\lambda_n} a_n^2$$

and

$$(G - G^0, G - G^0) = \sum a_n^2 .$$

If, now, we can say that a_n is real for all n, it follows that

$$- \frac{1}{\lambda_+} (G - G^0, G - G^0) \leq (g^0 - G^0, G - G^0) \leq - \frac{1}{\lambda_-} (G - G^0, G - G^0)$$

where λ_+ is the smallest positive λ_n and λ_- is the negative λ_n of smallest absolute value. (Note that λ_n cannot equal zero for any n.) Since G^0 is known in terms of ν' and G we have obtained an estimate for the error in using the first two terms of (23) as an approximation to ν. The accuracy of G^0 as an approximation to g^0 may also be obtained from

$$(g^o - G^o, g^o - G^o) = \sum b_n^2$$

$$\leq \max \left(\frac{1}{\lambda_+}, -\frac{1}{\lambda_-} \right) (G - G^o, G - G^o) .$$

This enables us to determine the error in quantities such as (f', g^o).

In practice this method may not be at all easy to apply because of the many restrictions. We have first to verify, in any particular problem, that the ϕ's satisfy the conditions imposed and then that G and ν' can be chosen so that a_n is real. (The latter condition is, at any rate, complied with when f and L are real.) Then λ_+ and λ_- (or bounds for them) must be determined. Fortunately, great accuracy is not required here because we make $(G - G^o, G - G^o)$ as small as possible by taking the best choice of ν' among those values which keep a_n real.

References

[1] Volterra, V. - Sopra un Problema die Elettrostatica; Il Nuovo Cimento, 16, 49-57 (1884).

[2] Levine, H. and Schwinger, J. - On the theory of diffraction by an aperture in an infinite plane screen. I.; Phys. Rev., 74, 958-74 (1948).

[3] Kohn, W. - Variational methods in nuclear collision problems; Phys. Rev., 74, 1763-72 (1948).

[4] Lippmann, B.A. - Variational formulation of a grating problem, Rept. N.D.A. 18-8 Nuclear Development Associates, Inc., White Plains, N.Y. (1952).

[5] Magnus, W. - Infinite matrices associated with diffraction by an aperture; Quart. App. Math., 11, 77-86 (1953). New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-32, May, 1951.

[6] Kato, T. - Upper and lower bounds of scattering phases; Progress of Theor. Phys., 6, 394-407 (1951).

Date Due

PRINTED IN U. S. A.

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

NYU
EM
78 Jones

AUTHOR

A critique of the varia-
TITLE tional method in scatter
ing problems.

NYU
EM
78

Jones

A critique of the variational
method in scattering problems.